A Simple Model for Antarctic Near-Surface Index of Refraction and Radio Pulse Trajectories

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Abstract

In this note, I derive the standard model for the index of refraction of Antarctic ice, as a function of ice depth. The model provides a good fit to the data, but does not account for small-scale fluctuations in snow density in the upper regions of the firn. The standard model implies curved paths for radio pulses. In specific situations, Fermat's principle implies that the *shadowing effect* should occur. Finally, the firn data suggests that although shadowing is possible, a more complete model should include surface propagation due to ray-trapping between local snow layers.

1 A Derivation of the Firn Model

Let the region of firn be described by N blocks labelled by j, with varying density ρ_j. Let the snow surface correspond to $j = N$, $z = 0$, and the beginning of solid ice (the bottom of the firn) correspond to $j = 0$, $z = h$. Each block has a height Δz, and a volume $v = A\Delta z$. Consider the normal force on block j:

$$F_j = \rho_j v g + \sum_{i=j+1}^{N} \rho_i v g \tag{1}$$

The normal force on block $j+1$ is

$$F_{j+1} = \rho_{j+1} v g + \sum_{i=j+2}^{N} \rho_i v g \tag{2}$$

Subtracting the normal forces gives

$$\frac{F_{j+1} - F_j}{g v \Delta z} = \frac{\rho_{j+1} - \rho_j}{\Delta z} + \frac{1}{\Delta z^2}\left(\Delta z \sum_{i=j+2}^{N} \rho_i - \Delta z \sum_{i=j+1}^{N} \rho_i\right) \tag{3}$$

Taking the limit $\Delta z \to 0$ yields

$$\frac{F'}{gv} = \rho' + \frac{1}{dz^2}\left(\int_{z_2}^{0} dz \rho(z) - \int_{z_1}^{0} \rho(z)\right) \tag{4}$$

$$\frac{F'}{gv} = \rho' - \frac{1}{dz^2}\int_{z_1}^{z_2} dz \rho(z) \tag{5}$$

In these steps, the height z_2 corresponds to block $j+1$, and z_1 corresponds to block j. Recall that $z_1 < z_2 < 0$. Note also that the derivative of the normal force approaches zero as fast as the volume element approaches zero, so the left-hand side is a constant. Taking the derivative of both sides gives

$$0 = \rho'' - \frac{A}{v}\left(\frac{\rho(z_2) - \rho(z_1)}{dz}\right) \tag{6}$$

$$\rho'' = \left(\frac{A}{v}\right)\rho' \tag{7}$$

It is given that the density must approach asymptotically a known, constant value. Thus, the second-order differential equation has a solution with two free parameters, ρ_1 and z_0, for $z < 0$:

$$\rho(z) = \rho_0 - \rho_1 e^{z/z_0} \tag{8}$$

Using the boundary conditions $\rho(0) = \rho_s$ (snow) and $\rho(z) = \rho_{ice}$ for $|z| \gg z_0$, $z < 0$, with $\Delta\rho = \rho_{ice} - \rho_s$, the final solution is

$$\rho(z) = \rho_{ice} - \Delta\rho e^{z/z_0} \tag{9}$$

For dielectric materials like snow and ice, the index of refraction is usually approximated as a linear equation of density: $n(z) \approx 1 + b\rho(z)$, and this is usually justified through expanding the Landau-Lifshitz-Looyenga equation (see below). Thus, the index versus depth of the metamorphosis from snow to ice follows a function like

$$n(z) = n_0 - n_1 e^{z/z_0} \tag{10}$$

At $z = 0$, $n(0) = n_s$ (snow), and as $|z| \gg z_0$, for $z < 0$, $n = n_{ice}$. Letting $\Delta n = n_{ice} - n_s$, the index equation becomes

$$\boxed{n(z) = n_{ice} - \Delta n e^{z/z_0}} \tag{11}$$

2 The Landau-Lifshitz-Looyenga Equation

Let the complex dielectric constants of snow and ice be ϵ_i and ϵ_s, and the dielectric constant of their mixture be $\epsilon(z)$. Further, let the two separate dielectrics each have volume fractions v_i and v_s, with volume fractions $v_i + v_s = 1$, $\rho(z) = v_i\rho_i + v_s\rho_s$. The Landau-Lifshitz-Looyenga equation gives

$$\epsilon(z) = \left(v_i\epsilon_i^{1/3} + v_s\epsilon_s^{1/3}\right)^3 \tag{12}$$

Let the real and imaginary parts of dielectric constants follow the notation $\epsilon = \epsilon' + i\epsilon''$, and defined the *loss tangent* as $\tan\delta = \epsilon''/\epsilon'$. Ice has a loss tangent of order 10^{-3} at RF frequencies, and the loss tangent of snow is smaller. To first order in $\tan\delta_i$ and $u = v_2/v_1 < 1$, with $\alpha = (\epsilon_2'/\epsilon_1')^{1/3}$, it may be shown that

$$\sqrt{\Re\epsilon(z)} = n(z) \approx v_i^{3/2}\epsilon_i'^{1/2}(1 + u\alpha) \tag{13}$$

Setting $v_2 = 0$, $v_1 = 1$ (or $u = 0$) reproduces the expected $n = \sqrt{\epsilon_i'}$ for pure ice. With $\beta = 3/2(\rho_s/\rho_i)$, $v_i^{3/2}$ is approximately

$$v_i^{3/2} \approx \left(\frac{\rho(z)}{\rho_i}\right)^{3/2} (1+u\beta)^{-1} \tag{14}$$

Combining equations, and recalling that $\epsilon_i'^{1/2} = n_{ice}$, the result is

$$\frac{n(z)}{n_{ice}} \approx \left(\frac{1+u\alpha}{1+u\beta}\right)\left(\frac{\rho(z)}{\rho_i}\right)^{3/2} \tag{15}$$

Expanding to first order about $\rho(z)/\rho_i = 1$:

$$\frac{n(z)}{n_{ice}} \approx -\frac{1}{2}\left(\frac{1+u\alpha}{1+u\beta}\right) + \frac{3}{2}\left(\frac{1+u\alpha}{1+u\beta}\right)\frac{\rho(z)}{\rho_i} \tag{16}$$

Using $n_s = \epsilon_s'^2 = 1.3$, $n_i = \epsilon_i'^2 = 1.78$, $\rho_s = 0.4$ g/cc, $\rho_i = 0.917$ g/cc, and $u = 0.1$ as an example, a linear equation for the index versus density near the firn/ice boundary layer is

$$\frac{n(z)}{n_{ice}} \approx -0.51 + 1.66\rho(z)[g/cc] \tag{17}$$

3 Fitting the Firn Model to the Data

Many analyses and derivations have been done to produce the index of refraction versus depth curve in different locations throughout Antarctica. Figure 1 contains a summary of such measurements. In the figure, the function $n(z) = A - B\exp(Cz)$ is fit to the data points. Where density data was available, the empirical conversion of $n(z) = 1.0 + 0.86\rho(z)$ has been used.

The value for A in all the fits was restricted to $n_{ice} = 1.78$, from the differential equation solution to the gravity-density problem. No restriction was placed on the value for $B = \Delta n$, but note that the results are close to $1.78 - 1.29 = 0.49$, where 1.29 is the expected value for n_s (Hanson 2013). Thus, the fits are all measuring n_s accurately. The slopes $C = z_0^{-1}$ differ across the Antarctic continent, and are statistically lower at the South Pole compared to other locations.

Table 1 summarizes the results for the fits to the points in the figure. The snow surface index of refraction is derived from the B parameter, assuming $A = n_{ice}$, and $B = \Delta n = n_{ice} - n_s$. These results may be compared to resuts from the upper 2 m ($n_s = 1.29 \pm 0.02$), obtained at the Ross Ice Shelf via multiple tecniques (Hanson 2013). The curves MB#1 and MB#2 refer to two cores drilled in Moore's Bay (Ross Ice Shelf) in 2016, and the references for the rest of the data may be found in the Table.

Ref./Location	$A = n_{ice}$	B	n_s	C (m^{-1})	z_0 (m)
MB#1/Moore's Bay	1.78	0.46 ± 0.01	1.32 ± 0.01	0.029 ± 0.002	34.5 ± 2
MB#2/Moore's Bay	1.78	0.481 ± 0.007	1.299 ± 0.007	0.027 ± 0.001	37 ± 1
Ebimuna (1983)/Byrd	1.78	0.464 ± 0.006	1.316 ± 0.006	0.0244 ± 0.0004	41 ± 1
Ebimuna (1983)/Mizuho	1.78	0.423 ± 0.008	1.357 ± 0.006	0.027 ± 0.001	37 ± 1
RICE (2004)/South Pole	1.78	0.43 ± 0.02	1.35 ± 0.02	0.014 ± 0.001	71 ± 5
Eisen (2003)/South Pole	1.78	0.48 ± 0.01	1.3 ± 0.01	0.020 ± 0.001	50 ± 2.5
Gow (xxxx)/South Pole	1.78	0.435 ± 0.01	1.345 ± 0.01	0.016 ± 0.001	62.5 ± 4

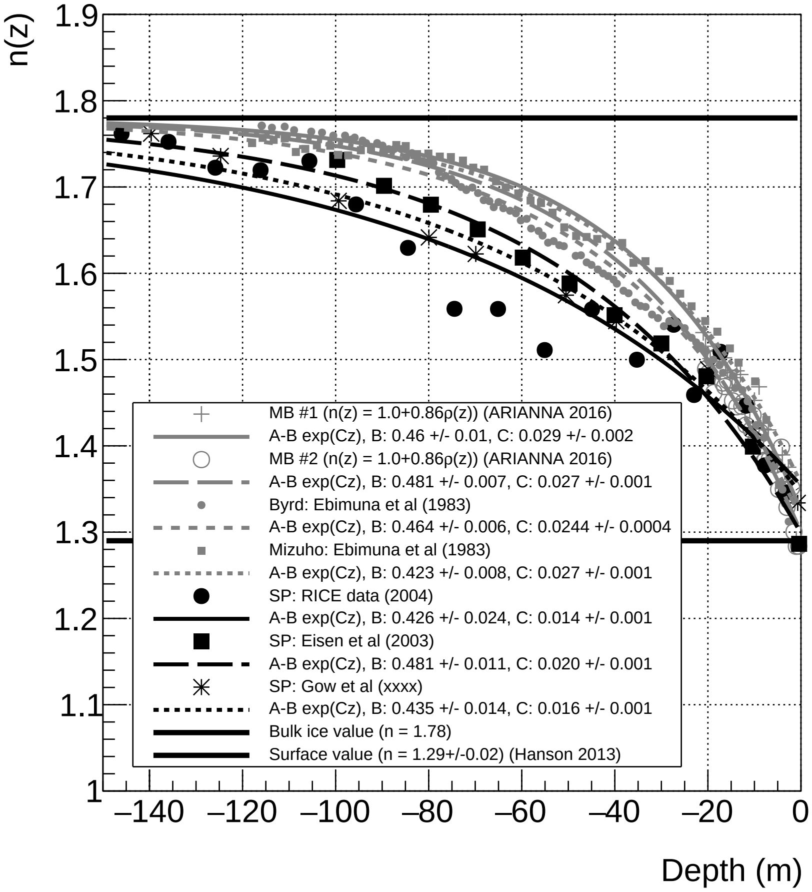

Figure 1: A summary of all the $n(z)$ data discovered for various locations in Antarctica, including Moore's Bay (MB) and the South Pole. All data points and fit lines that are black correspond to the South Pole, and the gray points and fit lines correspond to Moore's Bay, Byrd station, and Mizuho station.

4 Fermat's Principle, and Ray-Tracing

A key question for ARA/ARIANNA future designs is the expected path of a radio pulse from an Askaryan event in firn. Beginn with Fermat's principle, which states that a ray must traverse the path that minimizes the travel time. Fermat's principle is similar to the principle of least-action, in which a massive particle takes the path of least-resistance (cite Wiki Fermat's).

$$\delta S = 0 \tag{18}$$

$$\delta \int_A^B n(z)(1+\dot{y}^2)^{1/2} dx dy dz = \int_A^B L(z,\dot{y}) dx dy dz = 0 \tag{19}$$

Derivatives indicated by the dot notation are with respect to z, not time. The assumption that $x = \dot{x} = 0$ has been taken without loss of generality. Note that $\dot{y} = dy/dz$ is unit-less, and $\ddot{y}$ has units of inverse meters. Using the Euler-Lagrange equations to minimize the variation in the path, and letting $u = \dot{y}$:

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) - \left(\frac{\partial L}{\partial y}\right) = 0 \tag{20}$$

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \tag{21}$$

$$\dot{u} = -\left(\frac{\dot{n}}{n}\right)(u^3 + u) \tag{22}$$

Note that the units are inverse meters on each side of the equation: all factors of u are unit-less, and $\dot{n}$ has units of inverse meters. Putting in the model for $n(z)$, the final equation of motion is

$$\boxed{\dot{u} = z_0^{-1}\left(\frac{\Delta n e^{z/z_0}}{n_{ice} - \Delta n e^{z/z_0}}\right)(u^3 + u)} \tag{23}$$

As a check, note the deep ice limit: $|z| \gg z_0$, $z < 0$:

$$\dot{u} = 0 \tag{24}$$

The solution to this equation of motion, after solving for z is

$$z(y) = a + by \tag{25}$$

In other words, if the rays are far from the firn, the rays must propagate in straight lines. For the case of a shallow ray $z \to 0$, propagating initially with a horizontal velocity component satisfying $u^3 \gg u$, the main equation of motion reduces to

$$\frac{du}{dz} = \left(\frac{n_{ice} - n_s}{z_0 n_s}\right) u^3 \tag{26}$$

This is a variables-separable differential equation, with a solution

$$z(y) = -\frac{1}{2z_0}\left(\frac{n_{ice} - n_s}{n_s}\right) y^2 - y \tag{27}$$

Thus, for a very shallow ray, with initial horizontal velocity, the solution predicts shadowing, where a horizontal ray is eventually bent downward. Note that this does not imply yet that shadowing should happen in real Antarctic firn over an appreciably short distance, because the values for n_{ice}, n_s and z_0 have to be inserted. The next least-restrictive approximation for the shallow depth of the ray is $\exp z/z_0 \approx 1 + z/z_0$, rather than $z \to 0$. Let $q = \Delta n / z_0$. The full equation of motion, in this limit, reduces to

$$\frac{du}{dz} = z_0^{-1}\left(\frac{\Delta n + qz}{n_s - qz}\right) u^3 \tag{28}$$

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The final solution in this limit is

$$z(y) = \frac{Q_1}{8z_0}(y - y_1)^2 - \frac{Q_0}{Q_1} \tag{29}$$

Note that, in either the limit of $z \to 0$, or $\exp z/z_0 \approx 1 + z/z_0$, the solutions are quadratic, with curvature controlled by z_0^{-1}. That is, if z_0 increases, the concavity of the ray path, and thus, the level of shadowing, decreases. It is fascinating that the same snow metamorphosis that controls the compaction from snow to ice through gravity also controls the amount of ray bending, and that this number is measurable from the density variation versus depth.

Note: I will fill in tomorrow all the constants like Q_1 and Q_0. I have triple checked the units of everything. If the units don't seem to work, it's because the units are held in the constants.

5 Surface Propagation