My fun derivation of a differential equation for $\rho(z)$

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Abstract

I perform a short derivation of a differential equation describing the density of ice $\rho(z)$ as a function of depth z. It depends on the compressibility χ of ice, and we dig up some old data (well click on the top link from Google) from measurements of χ and its dependence on ρ and temperature T. We discuss the implications for the solution of our differential equation.

Keywords:

1. Derivation of Differential Equation

Starting with the layering like Jordan did, the index j identifies each layer, starting with the 0^{th} layer at the surface $z = 0$ and the N^{th} layer at depth $z = h$. The thickness of each layer is Δz and we consider an area A covered by each layer. The volume of each layer is $v = A\Delta z$.

Since the volume of each layer is the same, then

$$\frac{m_j}{\rho_j} = \frac{m_{j+1}}{\rho_{j+1}} \tag{1}$$

Rearranging, we get:

$$\rho_{j+1} = \rho_j \frac{m_{j+1}}{m_j} \tag{2}$$

The $j + 1^{\text{st}}$ layer has a bit more mass in the same volume because it's more compressed. Taking $m_{j+1} = m_j + \Delta m$,

$$\rho_{j+1} = \rho_j \frac{m_j + \Delta m}{m_j} = \rho_j \left(1 + \frac{\Delta m}{m_j}\right) \tag{3}$$

Rearranging, we get:

$$\rho_{j+1} - \rho_j = \rho_j \cdot \frac{\Delta m}{m_j} \tag{4}$$

Now using $\Delta\rho = \rho_{j+1} - \rho_j$ and $m_j = v\rho_j$,

$$\Delta\rho = \rho_j \cdot \frac{\Delta m}{m_j} \tag{5}$$

$$\frac{\Delta\rho}{\Delta m} = \frac{\rho_j}{m_j} = \frac{1}{v} \tag{6}$$

$$\Delta m = v\Delta\rho \tag{7}$$

Now we take the compressibility at a given layer χ_j, which is defined as:

$$\chi_j = -\frac{1}{v}\frac{\Delta v}{\Delta p}, \tag{8}$$

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relating a change in pressure to a corresponding change in volume. So then $\Delta v = -v\chi_j \Delta p$, and then we can express Δm as:

$$\Delta m = \rho_j \Delta v = v\, \rho_j\, \chi_j\, \Delta p \tag{9}$$

Note that we have dropped the minus sign because in the definition of χ_j, Δv would be a negative number (the volume gets smaller the more you squeeze it). But here, we're imagining compressing the volume of a layer v, and given that the layer gets compressed, Δv is how much volume we add to keep v the same. So Δv is positive.

Now, using $v = A\Delta z$ and $\Delta p = \Delta F/A$,

$$\Delta m = A\ \Delta z\, \rho_j\, \chi_j\, \frac{\Delta F}{A} = \Delta z\, \rho_j\, \chi_j\, \Delta F. \tag{10}$$

Now, following how Jordan started his calculation of the difference in force down on each layer:

$$F_j = \sum_{i=0}^{j} \rho_i v g \tag{11}$$

and

$$F_{j+1} = \sum_{i=0}^{j+1} \rho_i v g \tag{12}$$

so that

$$\Delta F = F_{j+1} - F_j = \rho_{j+1} v g \tag{13}$$

Don't look now but I'm switching out ρ_{j+1} for ρ_j!

$$\Delta F = F_{j+1} - F_j = \rho_j v g \tag{14}$$

Then, Equation 10 becomes:

$$\Delta m = \Delta z\, \rho_j^2\, \chi_j\ v\ g \tag{15}$$

Then using $\Delta m = v\Delta\rho$, we get:

$$v\ \Delta\rho = \Delta z\, \rho_j^2\, \chi_j\ v\ g \tag{16}$$

and fortunately our v's cancel, and we get:

$$\Delta\rho = \Delta z\, \rho_j^2\, \chi_j\ g \tag{17}$$

and so then replacing $\Delta\rho/\Delta z$ with $d\rho/dz$, we get our differential equation:

$$\frac{d\rho(z)}{dz} = \rho(z)^2\, \chi(z)\ \ g \tag{18}$$

where here we note that the compressibility χ could be a function of depth z.

2. Solutions

The solutions will depend on the form that we take for $\chi(z)$.

2.1. Constant $\chi(z)$

If we take $\chi(z) = \chi_0$ = a constant, then the solution takes the form:

$$\rho = c\frac{1}{z+a} \tag{19}$$

where c and a are constants to be determined in what follows. Differentiating, and using Equation **??**, we get:

$$\frac{d\rho}{dz} = -c\frac{1}{(z+a)^2} = c^2\left(\frac{1}{z+a}\right)^2 \chi_0\ g \tag{20}$$

and we find:

$$c = -\chi_0 \, g. \tag{21}$$

Now let's take the density at the surface to be ρ_0. Then,

$$\rho_0 = -\chi_0 \, g \frac{1}{a} \tag{22}$$

which gives:

$$a = -\frac{\chi_0 \, g}{\rho_0} \tag{23}$$

So, putting these together, Equation 19 becomes:

$$\rho = -\chi_0 \, g \frac{1}{z - \frac{\chi_0 g}{\rho_0}} \tag{24}$$

2.2. $\chi(z) \sim 1/\rho(z)$

Notice from Equation 18 that if we take $\chi(z) = A/\rho(z)$, then

$$\frac{d\rho(z)}{dz} = A\rho(z) \; g \tag{25}$$

and so the solution is of the form:

$$\rho = ae^{bz}. \tag{26}$$

So now differentiating and plugging into Equation 25, we get:

$$\frac{d\rho}{dz} = bae^{bz} = Aae^{bz} \, g, \tag{27}$$

so that $b = Ag$. So now our solution becomes:

$$\rho = ae^{Agz}. \tag{28}$$

and now we have to find a. Taking $\rho = \rho_1$ at $z = h$, we find

$$\rho_1 = ae^{Agh}, \tag{29}$$

and so $a = \rho_1 e^{-Agh}$, and our solution becomes:

$$\rho = \rho_1 e^{-Agh} e^{Agz} = \rho_1 e^{Ag(z-h)} \tag{30}$$

3. Data on Compressibility

I found an old paper where some folks published measurements of compressibility of packed snow as a function of initial density and temperature. Gunars Abele and Anthony J. Gow, Compressibility of Compacted Snow, CRREL Report 76-21, June 1976. Interestingly they found (I think) that for high enough initial density, the compressibility goes like $1/\rho$ or some decreasing power law of ρ. If $\chi \sim 1/\rho$ then for large enough depths where the density is high enough the density goes like an exponential as has been proposed, as in Jordan's writeup. For small enough densities, and I have to check where densities at the surface are the right size, then the ρ dependence goes like Equation 24 instead.

4. Acknowledgements

I started from Jordan's derivation and went a different direction. I look forward to arguing about which is correct!