The Units of the Discrete Fourier Transform

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Abstract

Here I write down carefully the dimensions and units of all components of a discrete Fourier Transform and Parseval's Theorem.

1 Sources and Other Reading

I will take as my primary source for information on the discrete Fourier Transform to be *Numerical Recipes: The Art of Computing, 3rd Edition* by Press *et al.* All equations I cite come from Chapters 12-13 unless stated otherwise. I will often cite them as NR-12.0.1, which should be read "Numerical Recipes, Equation 12.0.1".

As a preface, I will refresh some terminology. When we make a measurement of a physical observable, that observable has both a dimension, and a units system to quantify measurements in that dimension. The *dimension* is a measure of a physical variable. For example, the physical observable "how far do I have to walk to get across the room" could be measured by examining the *dimension* known as length. If I want to quantify that measurement, I must pick a system of *units*, like feet. So the distance across the room has *dimension* length, and can be measure in *units* of feet, or meters, or furlongs, etc.

2 The Continuous Fourier Transform

2.1 The Continuous Fourier Pair

Generally, the continuous Fourier transform relates the continuous time domain representation of a function $h(t)$ to its continuous frequency domain representation $H(f)$. For us, $h(t)$ is almost always a waveform from a detector; $h(t)$ has dimension of voltage, and typically units of Volts (V). Time (t) is usually measured in seconds (s). Linear frequency (f) has dimensions of cycles per time, and typically has units of cycles per second, or Hertz (Hz = 1/s). To denote the units of a quantity, I will use square brackets [...].

To summarize, $[h(t)] = \text{V}$, $[t] = \text{s}$, $[f] = 1/\text{s} = \text{Hz}$.

Now, $h(t)$ is related to $H(f)$ by the following mathematical relationships, known as a *Fourier Pair*. It is given in NR-12.0.1 by:

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{-2\pi i f t}\, df \tag{1}$$

and

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f t}\, dt \tag{2}$$

Now, the units. Observe that the complex exponentials ($e^{\pm 2\pi i f t}$) are dimensionless. This is because $[f] \cdot [t] = \mathrm{Hz} \cdot \mathrm{s} = \frac{1}{\mathrm{s}} \cdot \mathrm{s} = 1$. The dimension of df is inverse time, so $[df] = 1/\mathrm{s}$. This means that $H(f)$ has units of $h(t) \cdot dt$ or V·s. In many fields, this is written as V/Hz. Again, $[H(f)] = \mathrm{V} \cdot \mathrm{s} = \mathrm{V/Hz}$.

2.2 The Continuous Parseval's Theorem

The power in a signal is the same in the time or the frequency domain. This leads to the continuous version of Parseval's Theorem:

$$\text{Power} = \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df \tag{3}$$

The units check out. $[h(t)^2 dt] = \mathrm{V}^2 \cdot \mathrm{s}$, and $[H(f)^2 df] = \mathrm{V}^2/\mathrm{Hz}^2 \cdot \mathrm{Hz} = \mathrm{V}^2/\mathrm{Hz} = \mathrm{V}^2 \cdot \mathrm{s}$. The quantity $h(t)^2$ does not expressly have the units of power (recall from intro physics that $P = V^2/R$). In this field, one assumes the voltages are being run through some imaginary impedance R. Because the power has units of $\mathrm{V}^2 \cdot \mathrm{s}$, it is often seen on plots as having units of V^2/Hz.

3 The Discrete Fourier Transform

As scientists, we are never able to observe a voltage source continuously; instead, we are stuck with discrete samples of a waveform. Analogous to the continuous cases above, there are discrete versions of the Fourier Pairs, and discrete versions of Parseval's theorem.

3.1 The Discrete Fourir Pair

Imagine that for a period of time T, you take data every Δ seconds, and end up with N samples of data. Let each of these samples have an index k; so $k = 1$ is the first sample, $k = 2$ is the second, and so on. Each sample then has a timestamp, $t_k = k\Delta$, and a voltage measurement, $h_k = h(t)|_{t=t_k} = h(t_k)$. Because we have discrete data, we can only produce information about a discrete number of frequency modes.[1] We define the frequency modes of interest to be $f_n = \frac{n}{N\Delta}$ for $n = -N/2...N/2$. [2]

So, we take the continuous case from above ($H(f)$) and produce a discrete estimate of

[1]You might argue that if I want more data points, I could merely interpolate the waveform. But note that if you interpolate to smaller Δ, you increase the number of data points N commensurately, as the total time is a constant; $T = N\Delta$.

[2]As pointed out in NR section 12.1.1, this choice of frequency range is made to correspond to the upper and lower bounds of the Nyquist frequency–a logical choice as all power contained in a waveform is aliased there. But discussing that here would take me too far afield.

the continuous Fourier transform $H(f_n)$

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f t}\, dt \Longrightarrow H(f_n) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f_n t}\, dt$$

We can now estimate the integral with a sum with $dt \to \Delta$, $f_n = \frac{n}{N\Delta}$. Then as in NR-12.1.6

$$H(f_n) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f_n t}\, dt \;\approx\; \sum_{k=0}^{N} h(t_k)\, e^{(2\pi i f_n t_k)}\, \Delta \;\approx\; \Delta \sum_{k=0}^{N} h_k\, e^{2\pi i k n/N} = \Delta\, H_n$$

where H_n has been defined in the last step as

$$H_n = \sum_{k=0}^{N} h_k\, e^{2\pi i k n/N} \tag{4}$$

The discrete time points can be returned by performing the discrete inverse Fourier transform, given by NR-12.1.9:

$$h_k = \frac{1}{N} \sum_{n=0}^{N} H_n e^{-2\pi i k n/N} \tag{5}$$

The factor of $1/N$ is needed to normalize the sum. The parallel between equations (4) and (5) should remind you of those between equations (1) and (2), save the factor of N. The Fourier Pair in questions is really h_k and H_n, *not* h_k and $H(f_n)$.

An important note on terminology is now necessary. H_n, *without multiplication by* Δ, is termed, perhaps poorly, the *Discrete Fourier Transform.* To be clear, there are three phrases floating around at this point. There is the true, continuous Fourier Transform $H(f)$. There is the discrete estimate of the continuous Fourier Transform, $H(f_n)$. And there is the Discrete Fourier Transform itself, H_n, which is the object belonging to the discrete Fourier Pair. The language is subtle, bordering on malicious, so great care must be taken in navigating the literature. The language problem wouldn't be so dangerous were it not for the dimensions/ units problem the vocabulary choice engenders. This will be particularly clear when we explore the discrete version of Parseval's Theorem below.

Now, the units. As with the continuous case, note that the exponential is unitless, and $[h_k] = \mathrm{V}$. Therefore, for $H(f_n)$, we have $[H(f_n)] = [h_k \Delta] = \mathrm{V} \cdot \mathrm{s} = \mathrm{V/Hz}$. However, note that H_n *does not* have the same units as $H(f_n)$!! Because $H(f_n)$ has its time units brought in by multiplication of the Δ, we infer that the units of H_n are simply Volts. To be explicit: $[H(f_n)] = \mathrm{V} \cdot \mathrm{s}$ and $[H_n] = \mathrm{V}$. This conclusion bears worth repeatig in words: the units of the discrete estimate of the continuous Fourier transform ($H(f_n)$) *are not the same* as those of the Discrete Fourier Transform itself (H_n). $[H(f_n)] \neq [H_n]$.

3.2 The Discrete Parseval's Theorem

Parseval's theorem is still true: the power in a signal must be the same whether we write it down in frequency or time space. The discrete version of Perseval's theorem states:

$$\sum_{k=0}^{N} |h_k|^2 = \frac{1}{N} \sum_{n=0}^{N} |H_n|^2 \tag{6}$$

Note the units. The dimension of h_k^2 is voltage squared, and $[h_k^2] = \mathrm{V}^2$. This is also true for H_n; as we established above, $[H_n] = \mathrm{V}$ so $[H_n^2] = \mathrm{V}^2$. So $[h_k^2] = [H_n^2] = [V^2]$. The units check out, but pay attention to the quantity in the equality. It is *not* the discrete estimate of the continuous Fourier Transform $H(f_n)$, but rather the Discrete Fourier Transform itself H_n. The units would be *expressly incorrect* if we had used $H(f_n)$.

Many resources are not careful about stating the difference between $H(f_n)$ and H_n, especially when you move on to computing more complex things like the cross-correlations and Power Spectral Densities. But the difference is important from a physical standpoint, and forgetting it will mean the wrong answer, even if the code compiles. Books are also often not careful about pointing out the factor of $1/N$. When in doubt, always check Parseval's theorem manually, starting from the one fact that will always be unambiguous: the units of h_k are Volts, and the units of Δ are seconds. Derive everything else from there.

4 Summary Table

Symbol	Units	Meaning
$h(t)$	Volts (V)	Continuous, time domain representation of a function.
$H(f)$	Volts $\cdot$ Time or Volts/Hz (V/Hz)	Continuous, frequency domain representation of a function.
$H(f_n)$	Volts $\cdot$ Time or Volts/Hz (V/Hz)	Discrete estimate of the continous, frequency domain representation of a function
H_n	Volts (V)	Discrete Fourier Transform